

CHARACTERIZATION OF n -DIMENSIONAL NORMAL AFFINE SL_n -VARIETIES

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ABSTRACT. We show that any normal irreducible affine n -dimensional SL_n -variety X is determined by its automorphism group in the category of normal irreducible affine varieties: if Y is an irreducible affine normal algebraic variety such that $\mathrm{Aut}(X) \cong \mathrm{Aut}(Y)$ as ind-groups, then $Y \cong X$ as varieties. If we drop the condition of normality on Y , then X is not uniquely determined and we classify all such varieties. In case $n \geq 3$, all the above results hold true if we replace $\mathrm{Aut}(X)$ by $U(X)$, where $U(X)$ is the subgroup of $\mathrm{Aut}(X)$ generated by all one-dimensional unipotent subgroups. In dimension 2 we have some very interesting exceptions.

1. INTRODUCTION AND MAIN RESULTS

Our base field is the field of complex numbers \mathbb{C} . For an affine variety X the automorphism group $\mathrm{Aut}(X)$ has the structure of an ind-group. We will shortly recall the basic definitions and results in Section 2. The classical example is $\mathrm{Aut}(\mathbb{A}^n)$, $n > 1$, the group of automorphisms of the affine n -space \mathbb{A}^n . Recently, HANSPETER KRAFT proved the following result which shows that the affine n -space is determined by its automorphism group (see [Kr15]).

Theorem 0. *Let Y be a connected affine variety. If $\mathrm{Aut}(Y) \cong \mathrm{Aut}(\mathbb{A}^n)$ as ind-groups, then $Y \cong \mathbb{A}^n$ as varieties.*

In this paper we prove a similar result for some other varieties which we are going to define now. Let $d > 1$. Consider the action of $\mu_d = \{\xi \in \mathbb{C}^* | \xi^d = 1\}$ on \mathbb{A}^n by scalar multiplication and denote by $\pi : \mathbb{A}^n \rightarrow A_{d,n} := \mathbb{A}^n / \mu_d$ the quotient. This means that $A_{d,n}$ is an affine variety with coordinate ring $\mathcal{O}(A_{d,n}) = \mathbb{C}[x_1, \dots, x_n]^{\mu_d}$, the algebra of invariants (see [Mu74]). Note that $A_{d,n}$ is indeed an orbit space, because μ_d is finite. For $d > 1$, $A_{d,n}$ has an isolated singularity in $\pi(0)$ and π induces an étale covering $\mathbb{A}^n \setminus \{0\} \rightarrow A_{d,n} \setminus \{p(0)\}$ with Galois group μ_d . Later on we consider only the case $d > 1$.

Theorem 1. *Let X be a normal affine variety such that $\mathrm{Aut}(X) \cong \mathrm{Aut}(A_{d,n})$ as ind-group, then we have an isomorphism $X \cong A_{d,n}$ as varieties.*

The standard representation of SL_n on \mathbb{C}^n induces an action of SL_n on $A_{d,n}$ for any d , and we have the following result (see [KRZ17]).

Proposition 1. *Let $n \geq 3$, and let Y be an affine normal variety of dimension n with a non-trivial SL_n -action. Then Y is SL_n -isomorphic to $A_{d,n}$ for some $d \geq 1$.*

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Now we drop the assumption of normality. Note that the ring of regular functions $\mathcal{O}(A_{d,n})$ equals $\bigoplus_{k=0}^{\infty} \mathbb{C}[x_1, \dots, x_n]_{dk}$, where $\mathbb{C}[x_1, \dots, x_n]_{dk}$ denotes the homogeneous polynomials of degree dk . Consider the affine variety $A_{d,n}^s$ with coordinate ring $\mathcal{O}(A_{d,n}^s) = \mathbb{C} \oplus \bigoplus_{k=s}^{\infty} \mathbb{C}[x_1, \dots, x_n]_{dk} \subset \mathcal{O}(A_{d,n})$, $s \geq 1$. Then the induced morphism $\eta : A_{d,n} \rightarrow A_{d,n}^s$ is the normalization and has the property that the induced map $\eta' : A_{d,n} \setminus \{\star\} \xrightarrow{\sim} A_{d,n}^s \setminus \{\star\}$ is an isomorphism, where \star denotes the points corresponding to the homogeneous maximal ideals. In fact, η is SL_n -equivariant, and $A_{d,n} \setminus \{\star\}$ is an SL_n -orbit. We prove the following result.

Theorem 2. *Let X be an irreducible affine variety such that $\mathrm{Aut}(X)$ and $\mathrm{Aut}(A_{d,n})$ are isomorphic as ind-groups, then $X \cong A_{d,n}^s$ as a variety for some $s \in \mathbb{N}$.*

For $n = 2$, any irreducible affine normal variety X endowed with a non-trivial SL_2 -action is SL_2 -isomorphic to $A_{d,2}$, SL_2/T or $\mathrm{SL}_2/N(T)$ (see [Pop73]), where T is the standard subtorus of SL_2 and $N(T)$ denotes the normalizer of T .

Theorem 3. *Let X be an irreducible variety such that $\mathrm{Aut}(X) \cong \mathrm{Aut}(\mathrm{SL}_2/T)$ respectively $\mathrm{Aut}(X) \cong \mathrm{Aut}(\mathrm{SL}_2/N(T))$ as ind-groups, then $X \cong \mathrm{SL}_2/T$ respectively $X \cong \mathrm{SL}_2/N(T)$ as varieties.*

For an affine variety X we denote by $U(X) \subset \mathrm{Aut}(X)$ the subgroup generated by the one-dimensional unipotent subgroups. We do not know whether $U(X)$ has the structure of an ind-subgroup (i.e. whether $U(X) \subset \mathrm{Aut}(X)$ is closed). That is why we introduce the definition of an "algebraic homomorphism". This is a homomorphism $\phi : U(X) \rightarrow U(Y)$ such that for any subgroup $U \subset U(X)$, where U is a closed one-dimensional unipotent subgroup of $\mathrm{Aut}(X)$, the image $\phi(U) \subset \mathrm{Aut}(Y)$ is a closed one-dimensional unipotent subgroup and $\phi|_U : U \rightarrow \phi(U)$ is an isomorphism of algebraic groups.

Theorem 4. *Let $n > 2$ and let X be an irreducible affine variety. If there is a bijective algebraic homomorphism $U(X) \rightarrow U(A_{d,n})$, then $X \cong A_{d,n}^s$ for some $s \geq 1$.*

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2. PRELIMINARIES

The notion of an ind-group goes back to Shafarevich who called such objects *infinite dimensional groups*, (see [Sh66]). We refer to [Kum02] and [Kr15] for basic notions in this context.

Definition 1. By an *ind-variety* we mean a set V together with an ascending filtration $V_0 \subset V_1 \subset V_2 \subset \dots \subset V$ such that the following holds:

- (1) $V = \bigcup_{k \in \mathbb{N}} V_k$;
- (2) each V_k has the structure of an algebraic variety;
- (3) for all $k \in \mathbb{N}$ the subset $V_k \subset V_{k+1}$ is closed in the Zariski-topology.

A morphism from an ind-variety $V = \bigcup_k V_k$ to an ind-variety $W = \bigcup_m W_m$ is a map $\phi : V \rightarrow W$ such that for any k there is an m such that $\phi(V_k) \subset W_m$ and such

that the induced map $V_k \rightarrow W_m$ is a morphism of algebraic varieties. *Isomorphisms* of ind-varieties are defined in the obvious way.

Two filtrations $V = \bigcup_{k \in N} V_k$ and $V = \bigcup_{k \in N} V'_k$ are called *equivalent* if for every k there is an m such that $V_k \subset V'_m$ is a closed subvariety as well as $V'_k \subset V_m$. Equivalently, the identity map $\mathrm{id} : V = \bigcup_{k \in N} V_k \rightarrow V = \bigcup_{k \in N} V'_k$ is an isomorphism of ind-varieties.

An ind-variety V has a natural topology: a subset $S \subset V$ is open, (resp. closed), if $S_k := S \cap V_k \subset V_k$ is open, (resp. closed), for all k . Naturally, a locally closed subset $S \subset V$ has a natural structure of an ind-variety. It is called an *ind-subvariety*. An ind-variety V is called *affine* if all varieties V_k are affine. Throughout this paper we consider only affine ind-varieties and for simplicity we call them just ind-varieties.

The product of two ind-varieties is defined in the natural way. This allows to give the following definition.

Definition 2. An ind-variety G is said to be an *ind-group* if the underlying set G is a group such that the map $G \times G \rightarrow G$, $(g, h) \mapsto gh^{-1}$, is a morphism.

An ind-group G is called *connected* if for every $g \in G$ there is an irreducible curve C and a morphism $C \rightarrow G$ whose image contains the neutral element e and g .

A closed subgroup H of G (i.e. H is a subgroup of G and is a closed subset) is again an ind-group under the closed ind-subvariety structure on G . A closed subgroup H of an ind-group G is an algebraic group if and only if H is an algebraic subset of G .

The proof of the next result can be found in [St13] (see also [FK17]).

Proposition 2. Let X be an affine variety. Then $\mathrm{Aut}(X)$ has a natural structure of an affine ind-group.

Note that in [St13] one can also find the description of the ind-group structure on $\mathrm{Aut}(X)$.

3. AUTOMORPHISMS

Proposition 3. Any automorphism of $A_{d,n}$ lifts to an automorphism of \mathbb{C}^n .

Proof. Let $\phi \in \mathrm{Aut}(A_{d,n})$. First we claim that $p_i := \phi^*(x_i^d)$ and $p_j := \phi^*(x_j^d)$ are coprime in $\mathbb{C}[x_1, \dots, x_n]$, where $i \neq j$ and ϕ^* is the pull-back of ϕ . Let p be a common factor of p_i and p_j . Then $\tilde{p} := \prod_{g \in \mu_d} gp$ divides p_i^d and p_j^d . By construction it is clear that $\tilde{p} \in \mathcal{O}(A_{d,n})$, then $\phi^{-1}(\tilde{p})$ is a common factor of $(\phi^*)^{-1}(p_i^d) = x_i^{d^2}$ and $(\phi^*)^{-1}(p_j^d) = x_j^{d^2}$. Hence, $\tilde{p} \in \mathbb{C}$ and therefore, $p \in \mathbb{C}$.

We have $\phi^*(x_i^d)\phi^*((x_j^d)^{d-1}) = \phi^*(x_i^d x_j^{d(d-1)}) = \phi^*(x_i x_j^{d-1})^d$ i.e. $p_i p_j^{d-1} = q^d$ for some $q \in \mathcal{O}(A_{d,n})$. Because p_i is coprime with p_j , it follows that $p_i = q_i^d$ for some $q_i \in \mathbb{C}[x_1, \dots, x_n]$.

The map ϕ induces an automorphism of $A_{d,n} \setminus \{\pi(0)\}$ and we call it also by ϕ . Recall that the quotient $\pi : \mathbb{A}^n \rightarrow A_{d,n}$ induces an étale covering $\tilde{\pi} : \mathbb{A}^n \setminus \{0\} \rightarrow A_{d,n} \setminus \{\pi(0)\}$. As $\mathbb{A}^n \setminus \{0\}$ is simply connected, it follows that every continuous automorphism of $A_{d,n} \setminus \{\pi(0)\}$ can be lifted to a continuous automorphism of $\mathbb{A}^n \setminus \{0\}$. Since both varieties are complex manifolds and the covering is étale, the lift of a holomorphic automorphism is also holomorphic. Thus, the automorphism ϕ of $A_{d,n} \setminus \{\pi(0)\}$ lifts to a holomorphic automorphism ψ of $\mathbb{A}^n \setminus \{0\}$. Now consider

$q_i := \psi^*(x_i)$. This is a holomorphic function on $\mathbb{A}^n \setminus \{0\}$ with the property that $q_i^d = \psi^*(x_i^d) = \phi^*(x_i^d)$ is a polynomial. It follows that the meromorphic function $r_i := \frac{q_i}{p_i}$ is holomorphic outside the zero set of p_i and satisfies $r_i^d = 1$. This implies that r_i is a constant, hence $q_i = \omega_i p_i$ for some d -th root of unity ω_i , first outside the zero set of p_i and then everywhere. Thus $\psi^*(\mathbb{C}[x_1, \dots, x_n]) \subset \mathbb{C}[x_1, \dots, x_n]$ which means that ψ is an algebraic morphism $\mathbb{A}^n \rightarrow \mathbb{A}^n$. It is an isomorphism because ψ is bijective. \square

Let X be an affine variety, H be a finite group acting on X and let $\pi : X \rightarrow X/H$ be the quotient morphism. Denote by $\text{Aut}^H(X) \subset \text{Aut}(X)$ the subgroup of all automorphisms of X which commute with the image of H in $\text{Aut}(X)$.

Lemma 1. (a) $\text{Aut}^H(X) \subset \text{Aut}(X)$ is a closed ind-subgroup,
 (b) there is a canonical homomorphism of ind-groups $\phi : \text{Aut}^H(X) \rightarrow \text{Aut}(X/H)$,
 (c) if X is normal and contains only finitely many fixed points of H then every \mathbb{C}^+ -action on X/H lifts to a \mathbb{C}^+ -action on X .

Proof. (a) Consider the homomorphisms $\phi_h : \text{Aut}(X) \rightarrow \text{Aut}(X)$, $\phi_h(g) = ghg^{-1}$. Then $\text{Aut}^H(X) = \bigcap_{h \in H} \phi_h^{-1}(H)$, where $\phi_h^{-1}(H) \subset \text{Aut}(X)$ is a closed subvariety. This proves the claim.

(b) Now let $h \in H$, $f \in \mathcal{O}(X)^H$ and $\phi \in \text{Aut}^H(X)$. Then $\phi^* : \mathcal{O}(X) \xrightarrow{\sim} \mathcal{O}(X)$ is an isomorphism and $h(\phi^*(f)) = \phi^*((\phi^*)^{-1} \circ h \circ \phi^*)(f) = (\phi^* \circ h')(f) = \phi^*(f)$ for some $h' \in H$. Therefore $\phi^*(f) \in \mathcal{O}(X)^H$, which means that ϕ induces an automorphism of X/H .

(c) There is an isomorphism of the space of derivations $\text{Der}(\mathcal{O}(X))$ with $\text{Hom}(\Omega_X^1, \mathcal{O}(X))$, where Ω_X^1 denotes the Kähler differential forms on X . By [Ha80, Corollary 1.2], $\text{Hom}(\Omega_X^1, \mathcal{O}(X))$ is a reflexive sheaf. Hence, $\text{Hom}(\Omega_{X \setminus Y}^1, \mathcal{O}(X \setminus Y))$ coincides with $\text{Hom}(\Omega_X^1, \mathcal{O}(X))$ for any closed subset $Y \subset X$ of codimension at least 2 (see [Ha80, Proposition 1.6]). Since X is normal, the quotient X/H is normal too. This implies that $\text{Der}(\mathcal{O}(X/H)) = \text{Der}(\mathcal{O}(X/H \setminus Z))$ for any closed subset $Z \subset X/H$ such that $\text{codim}_{X/H}(Z) \geq 2$.

Let $Z \subset X/H$ be the image of the union of the set of fixed points under the action of the group H and the set of singular points of X . The map $\pi|_{X \setminus \pi^{-1}(Z)} : X \setminus \pi^{-1}(Z) \rightarrow X/H \setminus Z$ is a finite étale covering with group H . Hence, the pullback $\pi^*(T_{X/H \setminus Z})$ of the tangent bundle $T_{X/H \setminus Z}$ of $X/H \setminus Z$ coincides with $T_{X \setminus \pi^{-1}(Z)}$ and then $T_{X/H \setminus Z} = \pi_*^H(T_{X \setminus \pi^{-1}(Z)})$ which consists of H -invariant sections $X \setminus \pi^{-1}(Z) \rightarrow T_{X \setminus \pi^{-1}(Z)}$. This implies that $\text{Der}(\mathcal{O}(X/H)) = \text{Der}(\mathcal{O}(X/H \setminus Z))$ is naturally isomorphic to $\text{Der}^H(\mathcal{O}(X \setminus \pi^{-1}(Z))) = \text{Der}^H(\mathcal{O}(X))$, where $\text{Der}^H(\mathcal{O}(X)) \subset \text{Der}(\mathcal{O}(X))$ denotes the vector subspace of H -invariant derivations. This means that each derivation of $\mathcal{O}(X/H)$ lifts to a derivation of $\mathcal{O}(X)$ and then by [Vas69, Theorem 2.2], each locally nilpotent derivation of $\mathcal{O}(X/H)$ lifts to a locally nilpotent derivation of $\mathcal{O}(X)$. The claim follows. \square

Let us recall that a closed subgroup U of $\text{Aut}(X)$ is called a 1-dimensional unipotent subgroup if $U \cong \mathbb{C}^+$.

Proposition 4. The homomorphism $\phi_d : \text{Aut}^{\mu_d}(\mathbb{A}^n) \rightarrow \text{Aut}(A_{d,n})$ is surjective with kernel μ_d . Moreover, every 1-dimensional unipotent subgroup of $\text{Aut}(A_{d,n})$ is the image of some 1-dimensional unipotent subgroup of $\text{Aut}^{\mu_d}(\mathbb{A}^n)$.

Proof. The surjectivity of ϕ_d follows from Proposition 3. The last claim of the statement follows from Lemma 1 (c). What remains is to compute the kernel of ϕ_d .

It is clear that

$$\mathrm{Aut}^{\mu_d}(\mathbb{A}^n) = \{f = (f_1, \dots, f_n) \in \mathrm{Aut}(\mathbb{A}^n) \mid f_i \in \bigoplus_{k=0}^{\infty} \mathbb{C}[x_1, \dots, x_n]_{kd+1}, i = 1, \dots, n\}.$$

Now let $f = (f_1, \dots, f_n) \in \mathrm{Aut}^{\mu_d}(\mathbb{A}^n)$ be such that the map f' induced by f on \mathbb{A}^n/μ_d is the identity. This means that f' acts trivially on $\mathcal{O}(\mathbb{A}^n/\mu_d) = \mathbb{C} \oplus \bigoplus_{k \geq 1} \mathbb{C}[x_1, \dots, x_n]_{kd}$. Hence, $f'(x_i^d) = x_i^d$ for any i which implies that $f = (\xi_1 x_1, \dots, \xi_n x_n)$, where $\xi_i^d = 1$ for $i = 1, \dots, n$. In particular, $f'(x_i^{d-1} x_j) = x_i^{d-1} x_j$ which implies that $\xi_i^{d-1} \xi_j = 1$ for any i, j . Because $\xi_i^{d-1} \xi_i = 1$ we conclude that $\xi_i = \xi_j$. The claim follows. \square

4. ROOT SUBGROUPS

Let G be an ind-group, and let $T \subset G$ be a closed torus.

Definition 3. A closed subgroup $U \subset G$ isomorphic to \mathbb{C}^+ and normalized by T is called a root subgroup with respect to T . The character of T on $\mathrm{Lie} U \cong \mathbb{C}$ i.e. the algebraic action of T on $\mathrm{Lie} U$ is called the weight of U .

Let X be an affine variety and consider a nontrivial algebraic action of \mathbb{C}^+ on X , given by $\lambda : \mathbb{C}^+ \rightarrow \mathrm{Aut}(X)$. If $f \in \mathcal{O}(X)$ is \mathbb{C}^+ -invariant, then the *modification* $f \cdot \lambda$ of λ is defined in the following way:

$$(f \cdot \lambda)(s)x := \lambda(f(x)s)x$$

for $s \in \mathbb{C}$ and $x \in X$. It is easy to see that this is again a \mathbb{C}^+ -action. In fact, the corresponding locally nilpotent derivation to $f \cdot \lambda$ is $f\delta_\lambda$, where δ_λ is the locally nilpotent derivation which correspond to λ . It is clear that if X is irreducible and $f \neq 0$, then $f \cdot \lambda$ and λ have the same invariants. If $U \subset \mathrm{Aut}(X)$ is a closed subgroup isomorphic to \mathbb{C}^+ and if $f \in \mathcal{O}(X)^U$ is a U -invariant, then in a similar way we define the modification $f \cdot U$ of U . Choose an isomorphism $\lambda : \mathbb{C}^+ \rightarrow U$ and set $f \cdot U := \{(f \cdot \lambda)(s) \mid s \in \mathbb{C}^+\}$. Note that $\mathrm{Lie}(f \cdot U) = f \mathrm{Lie} U \subset \mathrm{Vec}(X)$.

If a torus T acts linearly and rationally on a vector space V , then we call V multiplicity-free if the weight spaces V_α are all of dimension ≤ 1 .

Lemma 2 ([Kr15]). *Let X be an irreducible affine variety and let $T \subset \mathrm{Aut}(X)$ be a torus. Assume that there exists a root subgroup $U \subset \mathrm{Aut}(X)$ with respect to T such that the T -module $\mathcal{O}(X)^U$ is multiplicity-free. Then $\dim T \leq \dim X \leq \dim T + 1$.*

5. A SPECIAL SUBGROUP OF $\mathrm{Aut}(X)$

For any affine variety X consider the normal subgroup $U(X)$ of $\mathrm{Aut}(X)$ generated by closed one-dimensional unipotent subgroups. The group $U(X)$ was introduced and studied in [AFK13], where the authors called it the group of special automorphisms of X . After [Kr15] we introduce the following notion of an algebraic homomorphism between these groups.

Definition 4. A homomorphism $\phi : U(X) \rightarrow U(Y)$ is algebraic if for any subgroup $U \subset U(X)$ such that $U \subset \mathrm{Aut}(X)$ is closed, $U \cong \mathbb{C}^+$, the image $\phi(U) \subset \mathrm{Aut}(Y)$ is closed and $\phi|_U : U \rightarrow \phi(U)$ is a homomorphism of algebraic groups. We say that $U(X)$ and $U(Y)$ are algebraically isomorphic, $U(X) \cong U(Y)$, if there exists a bijective homomorphism $\phi : U(X) \rightarrow U(Y)$ such that ϕ and ϕ^{-1} are both algebraic.

A subgroup $G \subset U(X)$ is called algebraic if $G \subset \text{Aut}(X)$ is the closed algebraic subgroup. The next lemma can be found in [Kr15, Lemma 4.2].

Lemma 3. *Let $\phi : U(X) \rightarrow U(Y)$ be an algebraic homomorphism. Then, for any algebraic subgroup $G \subset U(X)$ generated by one-dimensional unipotent subgroups of $\text{Aut}(X)$, the image $\phi(G)$ is an algebraic subgroup of $U(Y)$ and $\phi|_G : G \rightarrow \phi(G)$ is a homomorphism of algebraic groups.*

Lemma 4. *Let X be an irreducible affine variety, and let $\eta : \tilde{X} \rightarrow X$ be its normalization. Then every automorphism of X lifts uniquely to an automorphism of \tilde{X} and the induced map $\tilde{\eta} : U(X) \hookrightarrow U(\tilde{X})$ is an algebraic homomorphism.*

Proof. Let $\mathbb{C}(X)$ be the field of rational functions on X . Then any automorphism ϕ of the ring of regular functions $\mathcal{O}(X)$ uniquely extends to an automorphism ϕ' of $\mathbb{C}(X)$. We claim that $\mathcal{O}(\tilde{X})$ is invariant under ϕ' , which would prove the first part of the lemma. Indeed, by definition f belongs to $\mathcal{O}(\tilde{X})$ if there is a monic polynomial $F = t^n + c_1 t^{n-1} + \dots + c_n \in \mathcal{O}(X)[t]$ such that $F(f) = 0$. Then $\phi(F(f)) = G(\phi(f)) = 0$ for some monic $G \in \mathcal{O}(X)[t]$, which proves the claim.

To prove the second part of the lemma, we note that any action of an algebraic group G on X lifts uniquely to a G -action on \tilde{X} . This follows from the fact that $G \times \tilde{X}$ is normal, the universal property of normalization and the following diagram:

$$\begin{array}{ccc} G \times \tilde{X} & \longrightarrow & \tilde{X} \\ \downarrow \text{id}_G \times \eta & & \downarrow \eta \\ G \times X & \longrightarrow & X \end{array}$$

Therefore, each regular \mathbb{C}^+ -action on X lifts uniquely to a regular \mathbb{C}^+ -action on \tilde{X} , which proves the claim. \square

Proposition 5. *Let $n \geq 3$ and let X be an n -dimensional irreducible affine variety endowed with a non-trivial SL_n -action. Then $\mathcal{O}(X) = \mathbb{C} \oplus \sum_{i=1}^l \sum_{k=k_i}^\infty \mathbb{C}[x_1, \dots, x_n]_{kd_i}$ for some $l, k_i, d_i \in \mathbb{N}$. The same holds when $n = 2$ and the normalization of X is $A_{d,2}$ for some $d \in \mathbb{N}$.*

Proof. First, let $n \geq 3$. If X is normal, then by Proposition 1, $X \cong A_{d,n}$ for some $d \in \mathbb{N}$. It is clear that $\mathcal{O}(A_{d,n}) = \bigoplus_{k=0}^\infty \mathbb{C}[x_1, \dots, x_n]_{kd}$ is a direct sum of irreducible pairwise non-isomorphic SL_n -modules $\mathbb{C}[x_1, \dots, x_n]_{kd}$.

Now, consider any n -dimensional irreducible affine variety X endowed with a non-trivial SL_n -action and a normalization morphism $\eta : A_{d,n} \rightarrow X$. Since any SL_n -action on $\mathcal{O}(X)$ lifts to an SL_n -action on $\mathcal{O}(A_{d,n})$, it follows that $\mathcal{O}(X)$ is a SL_n -submodule of $\mathcal{O}(A_{d,n})$ and therefore $\mathcal{O}(X) = \bigoplus_{k \in \Omega} \mathbb{C}[x_1, \dots, x_n]_{kd}$, where Ω is a submonoid of \mathbb{N} under addition. Since $\mathcal{O}(X)$ is finitely generated, $\Omega \subset \mathbb{N}$ is a finitely generated submonoid i.e. there exist $k_1, \dots, k_l \in \mathbb{N}$ such that $\Omega = k_1\mathbb{N} + \dots + k_l\mathbb{N}$. The claim follows. \square

6. 2-DIMENSIONAL CASE

The next result can be found in [Pop73], §3 (see also [Kr84], §4).

Lemma 5. *Let X be an affine normal irreducible variety of dimension two endowed with a non-trivial SL_2 -action. Then X is SL_2 -isomorphic to one of the following varieties:*

- (a) $A_{d,2}$ for some $d \in \mathbb{N}$,
- (b) SL_2/T , where T is the standard subtorus of SL_2 ,
- (c) $\mathrm{SL}_2/N(T)$, where $N(T)$ is the normalizer of T .

The SL_2 -variety $A_{d,2}$ is the union of a fixed point and the orbit $(\mathbb{C}^2 \setminus \{0\})/\mu_d \cong \mathrm{SL}_2/U_d$, where μ_d acts by scalar multiplication on $\mathbb{C}^2 \setminus \{0\}$ and $U_d = \left\{ \begin{bmatrix} \xi & t \\ 0 & \xi^{-1} \end{bmatrix} \mid t \in \mathbb{C}, \xi \in \mathbb{C}^*, \xi^d = 1 \right\}$. Moreover, any closed subgroup of SL_2 of codimension ≤ 2 is either T , $N(T)$, U_d for some $d \geq 1$ or $B = \left\{ \begin{bmatrix} a & t \\ 0 & a^{-1} \end{bmatrix} \mid t \in \mathbb{C}, a \in \mathbb{C}^* \right\}$ (see [We52]).

The next result can be found in [Kr84, III.2.5, Folgerung 3].

Proposition 6. *If a reductive group G acts on an affine variety X and if the stabilizer of a point $x \in X$ contains a maximal torus, then the orbit Gx is closed.*

Proposition 7. *Let X be an SL_2 -variety and let $O = \mathrm{SL}_2 x$ be the orbit of x . Assume that $\dim O \leq 2$. Then we are in one of the following cases:*

- (a) x is a fixed point;
- (b) the orbit O is closed and SL_2 -isomorphic to SL_2/T or $\mathrm{SL}_2/N(T)$;
- (c) $\overline{O} = O \cup \{x_0\}$, where \overline{O} is the closure of the orbit O and x_0 is a fixed point. Moreover, either $\overline{O} \simeq \mathbb{A}^2$ or x_0 is an isolated singular point.

Proof. If the stabilizer of x contains a maximal torus then we are in case (a) or (b) by Proposition 6. Otherwise, from the classification of closed subgroups of SL_2 it follows that the stabilizer of x coincides with U_d for some $d \geq 1$ and \overline{O} does not contain orbits of dimension one. Hence, $\overline{O} = O \cup \{x_0\}$. It is clear that if \overline{O} is singular, then x_0 is an isolated singular point. If \overline{O} is smooth, then from Lemma 5 it follows that \overline{O} is isomorphic to \mathbb{A}^2 . \square

Note that $\mathrm{SL}_2/T \cong \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$, where Δ is the diagonal, and $\mathrm{SL}_2/N(T) \cong \mathbb{P}^2 \setminus C$, where C is a smooth conic (see [Pop73, Lemma 2]).

There is the following well-known result.

Lemma 6. *Let X be a variety and let $G \subset \mathrm{Aut}(X)$ be an algebraic subgroup. Assume that $X = Gx$ for $x \in X$. Then $\mathrm{Aut}^G(X) \cong N_G(Gx)/G_x$.*

In fact, the right-multiplications on G/H with elements from $N_G(H)/H$ are the automorphisms of G/H which commute with the left-multiplications with all elements from G .

Lemma 7. *Consider the natural SL_2 -action on $X = \mathrm{SL}_2/T, \mathrm{SL}_2/N$ or $A_{d,2}$, and denote by $S \subset \mathrm{Aut}(X)$ the image of SL_2 .*

(a) *If $X = \mathrm{SL}_2/T$, then $S \cong \mathrm{PSL}_2$ and $\mathrm{Aut}^S(X) = \{\tau, \mathrm{id}\}$. Moreover, τ acts freely on X , and $X/\tau \cong \mathrm{SL}_2/N(T)$.*

(b) *If $X = \mathrm{SL}_2/N(T)$, then $S \cong \mathrm{PSL}_2$ and $\mathrm{Aut}^S(X) = \mathrm{id}$.*

(c) *If $X = A_{d,2}$, then $S \cong \mathrm{SL}_2$ if d is odd and $S \cong \mathrm{PSL}_2$ if d is even. Moreover, $\mathrm{Aut}^S(X) \cong \mathbb{C}^*$ is given by the image of \mathbb{C}^* acting by scalar multiplication on \mathbb{A}^2 . In particular, the groups $\mathrm{Aut}(\mathrm{SL}_2/T)$ and $\mathrm{Aut}(\mathrm{SL}_2/N(T))$ are not isomorphic, and also not isomorphic to $\mathrm{Aut}(A_{d,2})$ for any $d \geq 1$.*

Proof. Since the natural action of SL_2 on SL_2/T or $\mathrm{SL}_2/N(T)$ is transitive, (a) and (b) are immediate consequences of Lemma 6. For (c) we remark that X contains the orbit $O \cong \mathrm{SL}_2/U_d$. For $d = 1$, i.e. for $X = \mathbb{A}^2$, the claim is well-known. If $d > 1$,

then $\text{Aut}(X) \cong \text{Aut}(O)$, since the complement of O in X is a singular point. Now the claim follows from Lemma 6. \square

The variety SL_2/T is isomorphic to the following so-called DANIELEWSKI surface, i.e. the smooth 2-dimensional affine quadric $V(xz - y^2 + y) \subset \mathbb{A}^3$ (see [DP09]) and the quotient map $\pi : \text{SL}_2 \rightarrow \text{SL}_2/T$ is given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow (ab, ad, cd)$. It is not difficult to see that $X := V(xz + y^2 - 1) \cong V(xz - y^2 + y) \subset \mathbb{A}^3$.

By Lemma 7, there is an automorphism $\tau \in \text{Aut}^S(X)$ which acts freely on X and the quotient $Y := X/\tau$ is isomorphic to $\text{SL}_2/N(T)$, i.e. $\pi : X \rightarrow Y$ is a principal $\mathbb{Z}/2$ -bundle. In particular, $\mathcal{O}(Y) = \mathcal{O}(X)^\tau$. An automorphism ϕ of X descends to an automorphism on Y if and only if ϕ sends τ -orbits to τ -orbits. In fact, such an automorphism sends τ -invariant functions of $\mathcal{O}(X)$ to τ -invariant functions of $\mathcal{O}(X)$. Since τ has order 2, this condition for ϕ is equivalent to the condition that ϕ commutes with τ . We first note that $\text{Aut}^\tau(X)$ is a closed subgroup of $\text{Aut}(X)$ and then the canonical map $p : \text{Aut}^\tau(X) \rightarrow \text{Aut}(Y)$ is a homomorphism of ind-groups. In fact, kernel of p equals $\langle \tau \rangle$.

The following proposition follows from Lemma 1(c).

Proposition 8. *Every \mathbb{C}^+ -action on Y lifts to a \mathbb{C}^+ -action on X . In particular, the image $p(\text{Aut}_\tau(X))$ contains $U(Y)$ and $p^{-1}(U(Y)) \subset U(X)$*

Corollary 1. *For every algebraic subgroup $G \subset U(Y)$ the inverse image $\pi^{-1}(G) \subset \text{Aut}_\tau(X)$ is algebraic. If G is generated by unipotent elements, then $\pi^{-1}(G) = \pi^{-1}(G)^0 \times \langle \tau \rangle$.*

By [Lam05, Theorem 6], $\text{Aut}(X)$ is the amalgamated product of the orthogonal group $\text{O}(3, \mathbb{C}) = \text{SO}(3, \mathbb{C}) \times \langle \tau \rangle$ and $J_T \rtimes \langle \tau \rangle$ along their intersection C_T , where $\tau = (-x, -y, -z)$, J_T is the group of automorphisms of the form

$$(x, y, z) \mapsto (\alpha x + 2\alpha y P(z) - \alpha z P^2(z), (y - z P(z)), \frac{1}{\alpha} z); \alpha \in \mathbb{C}^*, P \in \mathbb{C}[z].$$

Hence, $\text{Aut}(X)$ is generated by $U(X)$ and $\langle \tau \rangle$. Since $U(X)$ is the normal subgroup of $\text{Aut}(X)$, it follows that $\text{Aut}(X) = U(X) \rtimes \langle \tau \rangle$. By [Neu48, Corollary 8.11], $U(X)$ is the amalgamated product of $\text{SO}(3, \mathbb{C})$ and J_T . Note that the subgroup $U(X) = \text{Aut}^0(X) \subset \text{Aut}(X)$ is closed (see [Kr15, Lemma 6.3]), where $\text{Aut}^0(X)$ is the neutral component of $\text{Aut}(X)$. Hence, $U(X)$ is an ind-group. By the following computation

$$\begin{aligned} (tx, y, t^{-1}z) \circ (x + 2yP(z) - zP^2(z), (y - zP(z)), z) \circ (t^{-1}x, y, tz) = \\ = (x + 2ytP(tz) - zt^2P^2(tz), (y - ztP(tz)), z), \end{aligned}$$

it is easy to see that $U_i = \{(x + 2yP_i(z) - zP_i^2(z), (y - zP_i(z)), z) | P_i(z) = z^i\}$ is the root subgroup with weight $i + 1$ with respect to $T'' = \{(tx, y, t^{-1}z) | t \in \mathbb{C}^*\} \cong \mathbb{C}^*$ for any $i \in \mathbb{N} \cup \{0\}$. The fact that there is no other root subgroups with respect to T'' follows from amalgamated product structure.

Summarizing everything that is said above, we have the following result.

Proposition 9. *For $X = \text{SL}_2/T$ we have the following properties.*

- (a) *All closed subgroups $S \subset \text{Aut}(X)$ isomorphic to PSL_2 are conjugate.*
- (b) *The root subgroups with respect to a maximal torus T'' of some $S \cong \text{PSL}_2$ are multiplicity-free with weights 1, 2, 3, ...*

It is not difficult to see that $\mathrm{Aut}^\tau(X)$ is the amalgamated product of $\mathrm{SO}(3, \mathbb{C}) \times \langle \tau \rangle$ and $J^\tau \times \langle \tau \rangle$ along their intersection, where

$$J^\tau = \{(x, y, z) \mapsto (\alpha x + 2\alpha y P(z) - \alpha z P^2(z), (y - z P(z)), \frac{1}{\alpha} z); \alpha \in \mathbb{C}^*, P \in \bigoplus_{l=0}^{\infty} \mathbb{C} z^{2l}\}.$$

By [Neu48, Corollary 8.11], $\mathrm{Aut}^\tau(X)$ is the amalgamated product of $\mathrm{SO}(3, \mathbb{C}) \times \langle \tau \rangle$ and $J^\tau \times \langle \tau \rangle$ along their intersection.

Recall that map $p : \mathrm{Aut}^\tau(X) \twoheadrightarrow \mathrm{Aut}(Y)$ is the surjective homomorphism with kernel $\langle \tau \rangle$. Hence, $\mathrm{Aut}(Y) = \mathrm{Aut}^\tau(X) / \langle \tau \rangle$. By [Co63, Theorem 1], $\mathrm{Aut}(Y)$ is the amalgamated product of $\mathrm{SO}(3, \mathbb{C})$ and J^τ along their intersection. Therefore, $\mathrm{Aut}(Y) = U(Y)$.

Summarizing everything that is said above and Proposition 9, we have the following result.

Corollary 2. *The root subgroups with respect to a maximal torus T'' of any $S \cong \mathrm{PSL}_2$ are multiplicity-free with weights $1, 3, 5, \dots$. In particular, $U(\mathrm{SL}_2 / N(T)) \not\cong U(\mathrm{SL}_2 / T)$.*

Recall that by Corollary 4, there is a homomorphism $\phi_d : \mathrm{Aut}^{\mu_d}(\mathbb{A}^n) \rightarrow \mathrm{Aut}(A_{d,n})$ of ind-groups. Consider now the torus $T_n = \{(t_1, \dots, t_n) | t_i \in \mathbb{C}^*\} \subset \mathrm{Aut}(\mathbb{A}^n)$ and the torus $T'_n = \{(t_1, \dots, t_n) | t_i \in \mathbb{C}^*, t_1 \cdot \dots \cdot t_n = 1\} \subset U(\mathbb{A}^n)$ of dimension $n - 1$. Then $T_d := \phi_d(T'_n)$ is a maximal subtorus of $U(A_{d,n}) \subset \mathrm{Aut}(A_{d,n})$.

The following lemma is easy and follows from Lemma 12.

Lemma 8. *Let d be even. Then weights of root subgroups of $\mathrm{Aut}(A_{d,2})$ with respect to T_d are $\{\frac{kd+2}{2} | k \in \mathbb{N} \cup \{0\}\}$.*

By Jung - Van der Kulk theorem (see [Ju42] and [Kul53]) $\mathrm{Aut}(\mathbb{A}^2) = \mathrm{Aff}_2 *_{\mathbb{C}} J$, where Aff_2 is the group of affine transformations of \mathbb{A}^2 , $J = \{(ax+b, cy+f(x)) | a, c \in \mathbb{C}^*, b \in \mathbb{C}, f(y) \in \mathbb{C}[x]\}$ and $C = \mathrm{Aff}_2 \cap J$. Subgroup $\mathrm{Aut}^{\mu_k}(\mathbb{A}^2) \subset \mathrm{Aut}(\mathbb{A}^2)$ also has a structure of amalgamated product by [Neu48, Corollary 8.11], namely, $\mathrm{Aut}^{\mu_k}(\mathbb{A}^2)$ is the amalgamated product of GL_2 and $J_k = \{(ax+b, cy+f(x)) | a, c \in \mathbb{C}^*, b \in \mathbb{C}, f(y) \in \bigoplus_l \mathbb{C} x^{lk+1}\}$ along their intersection (see also [AZ13, Theorem 4.2]). From Proposition 4, it follows that $\mathrm{Aut}(A_{k,2}) \cong \mathrm{Aut}^{\mu_k}(\mathbb{A}^2) / \mu_k$ and by [Co63, Theorem 1], $\mathrm{Aut}^{\mu_k}(\mathbb{A}^2) / \mu_k$ is the amalgamated product of GL_2 / μ_k and J_k / μ_k along their intersection C_k . Hence, it is easy to see that $U(\mathbb{A}^2 / \mu_{2k})$ is the amalgamated product of PSL_2 and $J_{2k} = \{(ax+b, cy+f(x)) | a, c \in \mathbb{C}^*, b \in \mathbb{C}, f(y) \in \bigoplus_l \mathbb{C} x^{lk+1}\}$ along their intersection.

Note that $\mathcal{O}(A_{d,n}) \subset \mathbb{C}[x_1, \dots, x_n]$ for any $d \geq 1$. Hence, we can define the Jacobian matrix of $f = (f_1, \dots, f_n) \in \mathrm{Aut}(A_{d,n})$ in the usual way i.e. $\mathrm{Jac}(f) = (\frac{\partial f_i}{\partial x_j})_{i,j}$ and then $j(f) := \det \mathrm{Jac}(f)$. It is also well-known that $U(\mathbb{A}^2) = \{f \in \mathrm{Aut}(\mathbb{C}^2) | j(f) \in \mathbb{C}^*\}$. It follows that $U(A_{d,2}) = \{f \in \mathrm{Aut}(A_{d,2}) | j(f) \in \mathbb{C}^*\}$. Therefore, $U(A_{d,2}) \subset \mathrm{Aut}(A_{d,2})$ is the closed subgroup.

The following result was pointed to us by HANSPETER KRAFT.

Proposition 10. *Let Z be an irreducible affine normal variety of dimension 2.*

(a) *Assume that there is a bijective algebraic homomorphism $U(\mathrm{SL}_2 / T) \rightarrow U(Z)$. Then $Z \cong \mathrm{SL}_2 / T$ or $A_{2,2}$.*

(b) *Assume that there is a bijective algebraic homomorphism $U(\mathrm{SL}_2 / N(T)) \rightarrow U(Z)$. Then $Z \cong \mathrm{SL}_2 / N(T)$ or $A_{4,2}$.*

Proof. Choose an SL_2 -action on Z such that the the root subgroups with respect to the image $T \subset U(Z)$ of the diagonal torus $T \subset \mathrm{SL}_2$ are multiplicity-free with weights $1, 2, 3, \dots$. The existence of such an action is given by Proposition 9(b) for SL_2/T , and then follows for $\mathrm{SL}_2/N(T)$ by Corollary 2. By Lemma 5, Z is SL_2 -isomorphic to SL_2/T , to $\mathrm{SL}_2/N(T)$, or to $A_{d,2}$ for some $d \in \mathbb{N}$.

To prove the claim, we first note that $U(\mathrm{SL}_2/T) \not\cong U(\mathrm{SL}_2/N(T))$ by Corollary 2. Let $X \cong \mathrm{SL}_2/T$ or to $\mathrm{SL}_2/N(T)$. Then the isomorphism $U(X) \cong U(\mathbb{C}^2/\mu_d)$ implies that d is even by Lemma 7. By Lemma 13, weights of root subgroups of $U(X)$ and $U(A_{d,2})$ have to be equal and then Lemma 8 implies that $U(\mathrm{SL}_2/T)$ can only be isomorphic to $U(A_{2,2})$, and $U(\mathrm{SL}_2/N(T))$ can only be isomorphic to $U(A_{4,2})$ by Corollary 2.

To show that $U(A_{2,2})$ and $U(\mathrm{SL}_2/T)$ are algebraically isomorphic, we first note that the first factors from the amalgamated product (described above) of $U(A_{2,2})$ and $U(\mathrm{SL}_2/T)$ are isomorphic to PSL_2 . To show that J_2 and J_T are algebraically isomorphic, it is enough to say that they have the same weights with respect to the standart subtori. It remains to remark that $C_T \cong C_2$. Analogously, $U(A_{4,2})$ and $U(\mathrm{SL}_2/N(T))$ are algebraically isomorphic too. \square

7. HIGHER-DIMENSIONAL CASE

The next result can be found in [Lie11, Theorem 1]. Recall that by T'_n we denote the standard maximal subtorus of $\mathrm{SAut}(\mathbb{A}^n) = \{f = (f_1, \dots, f_n) \in \mathrm{Aut}(\mathbb{A}^n) \mid \mathrm{jac}(f) := \det[\frac{\partial f_i}{\partial x_j}]_{i,j} = 1\}$.

Lemma 9. *Let $U \subset \mathrm{SAut}(\mathbb{A}^n)$ be a one-dimensional unipotent subgroup. Then U is a root subgroup with respect to T'_n if and only if $U = U_\lambda = \{(x_1, \dots, x_i + cm_i, \dots, x_n) \mid c \in \mathbb{C}\}$, where $m_i = x_1^{\lambda_1} \dots x_{i-1}^{\lambda_{i-1}} x_{i+1}^{\lambda_{i+1}} \dots x_n^{\lambda_n}$. The character ξ_λ corresponding to the root subgroup U is the following: $\xi_\lambda : T'_n \rightarrow \mathbb{C}^*$, $t = (t_1, \dots, t_n) \mapsto t_i t_1^{-\lambda_1} \dots t_{i-1}^{-\lambda_{i-1}} t_{i+1}^{\lambda_{i+1}} \dots t_n^{\lambda_n}$.*

Remark 1. The last lemma can also be expressed in the following way (see [KS13, Remark 2]): there is a bijective correspondence between the T'_n -stable one-dimensional unipotent subgroups $U \subset \mathrm{Aut}(\mathbb{A}^n)$ and the characters of T'_n of the form $\lambda = \sum_j \lambda_j \epsilon_j$ where one λ_i equals 1 and the others are ≤ 0 . We will denote this set of characters by $X_u(T'_n)$:

$$X_u(T'_n) := \{\lambda = \sum \lambda_j \epsilon_j \mid \text{such that } \lambda_i = 1 \text{ and } \lambda_j \leq 0 \text{ for } j \neq i\}.$$

If $\lambda \in X_u(T'_n)$, then U_λ denotes the corresponding one-dimensional unipotent subgroup normalized by T'_n .

Lemma 10. *Consider the standard action of SL_n on $A_{d,l}$ and denote by $S_{n,d} \subset \mathrm{Aut}(A_{d,n})$ the image of SL_n . Then $S_{n,d} \cong \mathrm{SL}_n/\mu_{(n,d)}$, where (n,d) denotes the greatest common divisor of n and d . Moreover, $S_{n,d} \subset U(A_{d,n})$.*

Proof. By Proposition 4, there is a surjective homomorphism $\phi_d : \mathrm{Aut}^{\mu_d}(\mathbb{A}^n) \rightarrow \mathrm{Aut}(A_{d,n})$ of ind-groups with $\ker \phi = \mu_d$. Hence, $\mathrm{Aut}(A_{d,n}) \cong \mathrm{Aut}^{\mu_d}(\mathbb{A}^n)/\mu_d$ which shows that $S_{n,d} \cong \mathrm{SL}_n/\mu_d$. The second claim is clear. \square

Corollary 3. *If $U(A_{d,n})$ and $U(A_{l,n})$ are algebraically isomorphic, then $(d,n) = (l,n)$.*

Recall that by Proposition 4 there is a homomorphism $\phi_d : \mathrm{Aut}^{\mu_d}(\mathbb{A}^n) \rightarrow \mathrm{Aut}(A_{d,n})$ of ind-groups and we denote by T_d the subtorus $\phi_d(T'_n) \subset U(A_{d,n})$. Map ϕ_d induces the map $\tilde{\phi}_d : U^{\mu_d}(\mathbb{A}^n) \rightarrow U(A_{d,n})$ which has the kernel $\mu_{(n,d)}$, where $U^{\mu_d}(\mathbb{A}^n) \subset \mathrm{Aut}^{\mu_d}(\mathbb{A}^n)$ is a subgroup generated by \mathbb{C}^+ -actions.

In [BB67], it is proved that any faithful action of an $(n-1)$ -dimensional torus on \mathbb{A}^n is linear. This result is used in order to prove the following lemma.

Lemma 11. *Let T be an algebraic subtorus of $U(A_{d,n})$ of dimension $(n-1)$. Then there exists a bijective algebraic homomorphism $F : U(A_{d,n}) \xrightarrow{\sim} U(A_{d,n})$ such that $F(T) = T_d$.*

Proof. Torus $(\phi_d^{-1}(T))^0$ is an algebraic subgroup of $U(\mathbb{A}^n)$ isomorphic to $(\mathbb{C}^*)^{n-1}$. By [BB67, Theorem 1], the torus $\phi_d^{-1}(T)^\circ$ is conjugate to some subtorus \tilde{T} of T_n in $\mathrm{Aut}(\mathbb{A}^n)$. Since $U(\mathbb{A}^n)$ is the normal subgroup of $\mathrm{Aut}(\mathbb{A}^n)$, $\tilde{T} \subset T'_n = T_n \cap U(\mathbb{A}^n)$. Therefore, $(\phi_d^{-1}(T))^0$ is conjugate to T'_n which proves the claim. \square

Lemma 12. *Let $U \subset \mathrm{Aut}(A_{d,n})$ be a root subgroup with respect to T_d which has a character χ . Then U lifts to a root subgroup $\tilde{U} := (\phi_d^{-1}(U))^0 \subset \mathrm{Aut}_{\mu_d}(\mathbb{A}^n)$ with respect to $T'_n = (\phi_d^{-1}(T_d))^0$ with character $\tilde{\chi} := \psi^*(\chi)$ such that the following diagram*

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu_{(n,d)} & \longrightarrow & T'_n & \xrightarrow{\psi} & T_d \longrightarrow 1 \\ & & & & \downarrow \tilde{\chi} & & \downarrow \chi \\ & & & & \mathbb{C}^* & \xrightarrow{=} & \mathbb{C}^* \end{array}$$

commute, where $\psi = \phi_d|_{T'_n}$ and $\psi^*(\chi)$ is a pull-back of χ .

Proof. From Proposition 3 it follows that any root subgroup U of $\mathrm{Aut}(A_{d,n})$ with respect to T_d lifts to a unipotent subgroup $\tilde{U} = (\phi_d^{-1}(U))^0$ of $\mathrm{Aut}_{\mu_d}(\mathbb{A}^n)$. Moreover, \tilde{U} is normalized by $(\phi_d^{-1}(T_d))^\circ = T'_n$. Now, let $\tilde{u} \in \tilde{U}$ and $u = \phi_d(\tilde{u}) \in U$. Then $\phi_d(t^{-1} \circ \tilde{u}(s) \circ t) = \phi_d(\tilde{u}(t^k s)) = \tilde{u}(\psi(t^k)s)$ for some $k \in \mathbb{N}$, which proves the claim. \square

Proposition 11. *Let $X = A_{d,n}$, SL_2/T or $\mathrm{SL}_2/N(T)$ and Y be an irreducible affine variety. Let also assume that there is a bijective algebraic homomorphism $U(X) \xrightarrow{\sim} U(Y)$. Then $\dim Y \leq \dim X$. Moreover, if additionally Y is normal, then*

- (a) if $X \cong \mathrm{SL}_2/T$, then $Y \cong A_{2,2}$ or $Y \cong \mathrm{SL}_2/T$,
- (b) if $X \cong A_{2,2}$, then $Y \cong A_{2,2}$ or $Y \cong \mathrm{SL}_2/T$,
- (c) if $X \cong \mathrm{SL}_2/N(T)$, then $Y \cong A_{4,2}$ or $Y \cong \mathrm{SL}_2/N(T)$,
- (d) if $X \cong A_{4,2}$, then $Y \cong A_{4,2}$ or $Y \cong \mathrm{SL}_2/N(T)$,
- (e) otherwise, $Y \cong A_{d,n}$.

Proof. Fix an algebraic isomorphism $\psi : U(X) \xrightarrow{\sim} U(Y)$ and denote by T' the image of T_d if $X = A_{d,n}$ or the image of a maximal subtorus T of $U(X)$ if $X = \mathrm{SL}_2/T$ or $\mathrm{SL}_2/N(T)$. By Lemma 12, Proposition 9 and Corollary 2, all root subgroups $U \subset U(Y)$ with respect to T' have different weights. In particular, the root subgroups $\mathcal{O}(Y)^U \cdot U \subset U(Y)$ have different weights, which implies that $\mathcal{O}(Y)^U$ is multiplicity-free, because the map $\mathcal{O}(Y)^U \rightarrow \mathcal{O}(Y)^U \cdot U$ is injective. Hence, by Lemma 2, we have that $\dim Y \leq \dim T' + 1 = n$, which proves the first part of the lemma.

Now (a), (b), (c) and (d) follow from Proposition 10.

To prove (e), we note that $\mathrm{SL}_n/\mu_{(n,d)}$ belongs to $U(A_{d,n})$, which implies that SL_n acts non-trivially on Y and thus, by Proposition 1, $Y \cong A_{l,n}$ for some $l \in \mathbb{N}$. Hence, $\psi : U(A_{d,n}) \xrightarrow{\sim} U(A_{l,n})$. By Lemma 11 there exist an algebraic isomorphism $F : U(A_{l,n}) \xrightarrow{\sim} U(A_{l,n})$ such that $F(\psi(T_d)) = T_l$. Therefore, we can assume that $\psi(T_d) = T_l$. Groups $U(A_{l,n})$ and $U(A_{d,n})$ can be isomorphic only if $(n, d) = (n, l)$ by Corollary 3. Then, by Lemma 13, weights of root subgroups of $U(A_{d,n})$ and $U(A_{l,n})$ with respect to tori T_d and T_l respectively have to coincide and the claim follows from Lemma 12. \square

Proof of Theorem 1. It is clear from the definition that an isomorphism of ind-groups $\mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(A_{d,n})$ induces an algebraic isomorphism $U(X) \xrightarrow{\sim} U(A_{d,n})$. Now the claim follows from Proposition 11 and Lemma 7. \square

Let Z be an irreducible affine SL_n -variety of dimension $n \geq 2$ and $\psi : U(Z) \xrightarrow{\sim} U(A_{d,n})$ be an algebraic isomorphism. Let T be an $(n-1)$ -dimensional algebraic subtorus of $U(Z)$. Then, by Lemma 11, we can assume that $\psi(T) = T_d$.

Lemma 13. *Let $\psi : U(Z) \xrightarrow{\sim} U(A_{d,n})$ be as above. Then root subgroups U and $\psi(U)$ have the same weights with respect to T and T_d respectively.*

Proof. Let U be a root subgroup of $U(Z)$ with respect to T and $\mathrm{Lie} U = \mathbb{C}\nu$, where ν is a generator. Then $\psi(U)$ is the root subgroup of $U(A_{d,n})$ with respect to T_d . The algebraic isomorphism ψ induces an isomorphism $d\psi_e^u : \mathrm{Lie} U \rightarrow \mathrm{Lie} \psi(U)$. Note that action of T on U induces the action of T on $\mathrm{Lie} U$. Then $d\psi_e^u(t \circ \nu \circ t^{-1}) = d\psi_e^u(\chi(t)\nu) = \chi(\psi(t))\psi(\nu)$, where $\chi : T \rightarrow \mathbb{C}^*$ is a character. \square

Theorem 5. *Let $X = A_{d,n}$, SL_2/T or $\mathrm{SL}_2/N(T)$ and Y be an irreducible affine variety. Let also there is a bijective algebraic homomorphism $U(Y) \rightarrow U(X)$. Then*

- (a) *if $X \cong A_{2,2}$, then $Y \cong \mathrm{SL}_2/T$ or $Y \cong A_{2,2}^s$ for some $s \in \mathbb{N}$,*
- (b) *if $X \cong \mathrm{SL}_2/T$, then $Y \cong \mathrm{SL}_2/T$ or $Y \cong A_{2,2}^s$ for some $s \in \mathbb{N}$,*
- (c) *if $X \cong A_{4,2}$, then $Y \cong \mathrm{SL}_2/N(T)$ or $Y \cong A_{4,2}^s$ for some $s \in \mathbb{N}$,*
- (d) *if $n = 2$ and $X \cong \mathrm{SL}_2/N(T)$, then $Y \cong \mathrm{SL}_2/N(T)$ or $Y \cong A_{4,2}^s$ for some $s \in \mathbb{N}$,*
- (e) *otherwise, $Y \cong A_{d,n}^s$ for some $s \geq 1$.*

Proof. Let $\psi : U(X) \rightarrow U(Y)$ be an algebraic isomorphism. Proposition 11 implies that $\dim Y \leq \dim X$. Since SL_n acts regularly and non-trivially on X , SL_n also acts non-trivially and regularly on Y .

First, let X be isomorphic to $A_{d,n}$. Then by Lemma 5 and by Proposition 1, normalization of Y , which we denote by \tilde{Y} , is isomorphic to SL_2/T , $\mathrm{SL}_2/N(T)$ or $A_{l,n}$ for some $l \geq 1$.

First, assume that $\tilde{Y} \cong A_{l,n}$. Hence, Proposition 5 implies that $\mathcal{O}(Y) = \mathbb{C} \oplus \sum_{i=1}^r \sum_{k=k_i}^{\infty} \mathbb{C}[x_1, \dots, x_n]_{kl_i}$ for some $r, k_i, l_i \in \mathbb{N}$, $i \in \{1, \dots, l\}$. Let $\eta : A_{l,n} \rightarrow Y$ be the normalization morphism which by Lemma 4 induces the algebraic homomorphism $\tilde{\eta} : U(Y) \hookrightarrow U(A_{l,n})$. Note that $\mathrm{SL}_n/\mu_{(n,d)}$ acts faithfully on X . Then $\mathrm{SL}_n/\mu_{(n,d)}$ also acts faithfully on Y and therefore on $A_{l,n}$. This implies that $(n, d) = (n, l)$. By Lemma 11, we can assume without loss of generality that $\psi^{-1}(\tilde{\eta}^{-1}(T_l)) = T_d$.

It is clear that for any $s_i \geq k_i$, the group $U = \{(x_1 + cx_2^{s_i d_i + 1}, x_2, \dots, x_n) | c \in \mathbb{C}\} \subset \mathrm{Aut}^{\mu}(\mathbb{A}^n)$ induces a root subgroup \tilde{U} of $U(Y)$ with respect to $\tilde{\eta}^{-1}(T_l)$, and then U acts on $\mathcal{O}(Y)$. Since $(n, d) = (n, l)$, $\phi_d|_{T'_n}$ and $\phi_l|_{T'_n}$ have the same kernels,

and because \bar{U} and $\psi^{-1}(\bar{U})$ have the same weights with respect to $\tilde{\eta}^{-1}(T_l)$ and T_d respectively, by Lemma 12, U should also induce a \mathbb{C}^+ -action on $A_{d,n}$. Hence, U acts on $\mathcal{O}(A_{d,n})$ and then $d + s_i d_i \in \mathbb{N}d$. Since s_i is any natural number greater than or equal to k_i , $d|d_i$ for each i . Therefore, $\mathbb{N}d_1 + \dots + \mathbb{N}d_k \subset \mathbb{N}d$.

Analogously as above, for any $k \geq 1$, subgroup $U' = \{(x_1 + cx_2^{kd+1}, x_2, \dots, x_n) | c \in \mathbb{C}\} \subset \mathrm{Aut}^{\mu_d}(\mathbb{A}^n)$ induces a root subgroup of $U(A_{d,n})$ with respect T_d . Then U' acts on $\mathcal{O}(Y)$, which implies that $d_i k_i + kd \in (\mathbb{N}_{\geq k_1} d_1 + \dots + \mathbb{N}_{\geq k_l} d_l)$ for any i , where $\mathbb{N}_{\geq k} := \{m \in \mathbb{N} | m \geq k\}$. This shows that $\mathbb{N}_{\geq k_1} d_1 + \dots + \mathbb{N}_{\geq k_l} d_l = \mathbb{N}_{\geq \min_i \{k_i d_i | i=1, \dots, l\}} d$.

Now assume that \tilde{Y} is isomorphic to SL_2/T or to $\mathrm{SL}_2/N(T)$, then by Proposition 7, $Y = \tilde{Y}$. Then (e) follows from Proposition 10.

Let now $X \cong \mathrm{SL}_2/T$. Then by Lemma 5, \tilde{Y} can only be isomorphic to SL_2/T , $\mathrm{SL}_2/N(T)$ or $A_{2,2}$. By Proposition 10, \tilde{Y} is isomorphic to SL_2/T or to $A_{2,2}$. If $\tilde{Y} \cong \mathrm{SL}_2/T$, from Proposition 7, it follows that $Y = \tilde{Y}$. Hence, (b) follows from the first part of the proof. Analogously follows (d). \square

Proof of Theorem 2. The isomorphism $\mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(A_{d,n})$ induces an algebraic isomorphism $U(X) \rightarrow U(A_{d,n})$. Note that X admits a torus action of dimension n . From Theorem 5 it follows that X can only be isomorphic to $A_{d,n}^s$. Since normalization of $A_{d,n}^s$ is equal to $A_{d,n}$, it follows from [FK17] that there is a closed embedding $\mathrm{Aut}(A_{d,n}^s) \hookrightarrow \mathrm{Aut}(A_{d,n})$ of ind-groups and the proof follows from Lemma 4. \square

Proof of Theorem 3. Isomorphism $\mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(\mathrm{SL}_2/T)$ induces an algebraic isomorphism $U(X) \rightarrow U(\mathrm{SL}_2/T)$. Then the claim follows from Theorem 5 and Lemma 7. \square

Proof of Theorem 4. Follows from Theorem 5. \square

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